Behavior of Berenger's ABC for Evanescent Waves

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***Abstract*—A recently published ABC by Berenger has been shown to outperform all previous ABC's by several orders of magnitude. The cornerstone of this ABC is the so called Perfectly Matched Layer (PML). This layer absorbs electromagnetic waves incident at all angles without any reflection. In this letter we evaluate basic properties of the PML-medium for incident evanescent or inhomogeneous plane waves. We show that the evanescent wave is phase-shifted upon entering the PML-medium but retains its natural damping. Moreover, a substantial numerical reflection error must be taken into account, especially in the low frequency range. These results may be important in determining the number of PML-layers needed to obtain a given accuracy.**

I. Introduction

BERENGER [1], [2] has shown that the PML-medium behaves as an excellent absorber for incident plane waves at an arbitrary angle. Since the reflection coefficient for propagating waves is frequency independent, this suggests that the PML-ABC retains its efficiency in low frequency problems. This, however, ignores the fact that at low frequencies the evanescent wave part becomes more and more important. In this paper, we examine the behavior of the PML-medium for evanescent waves.

After this work was submitted, two papers on the same subject were presented at the ACES annual review [3], [4]. The results presented in [3] and [4] are in agreement with our conclusion. This letter provides more extensive technical derivations, that ,e.g., explain the cause of the total reflection from PML at low frequencies. The authors are grateful to the anonymous reviewer for bringing these papers to their attention.

II. Propagation of an Evanescent Wave in a PML Medium

Although similar results can be obtained for either two- as well as three-dimensional cases, we consider the two-dimensional TE-case for simplicity. In this case, the PML equations can be written as follows:

$$\varepsilon_0\frac{\partial E_x}{\partial t}+\sigma_y E_x=\frac{\partial(H_{zx}+H_{zy})}{\partial y} \tag{1a}$$

$$\varepsilon_0\frac{\partial E_y}{\partial t}+\sigma_x E_y=-\frac{\partial(H_{zx}+H_{zy})}{\partial x} \tag{1b}$$

$$\mu_0\frac{\partial H_{zx}}{\partial t}+\sigma_x^* H_{zx}=-\frac{\partial E_y}{\partial x} \tag{1c}$$

$$\mu_0\frac{\partial H_{zy}}{\partial t}+\sigma_y^* H_{zy}=\frac{\partial E_x}{\partial y}. \tag{1d}$$

Following the approach in [1], the field components of an evanescent plane wave are

$$[E_x\ E_y\ H_{zx}\ H_{zy}] = [(E_0\cosh\varphi)\ (jE_0\sinh\varphi)\ H_{zx0}\ H_{zy0}]e^{j\omega(t-\gamma_1 x-\gamma_2 y)}. \tag{2}$$

Given E_0 and φ, γ_1, and γ_2 can be determined from (1) and (2)

$$\gamma_1=\sqrt{\frac{\varepsilon_0\mu_0}{G}}\left(-\frac{\sigma_x}{\varepsilon_0\omega}-j\right)\sinh\varphi \tag{3a}$$

$$\gamma_2=\sqrt{\frac{\varepsilon_0\mu_0}{G}}\left(1-j\frac{\sigma_y}{\varepsilon_0\omega}\right)\cosh\varphi. \tag{3b}$$

This solution and its opposite correspond to waves that are exponentially decaying in the positive and negative x-direction, respectively. G is defined as

$$G=\sqrt{w_x\cosh\varphi^2+w_y\sinh\varphi^2} \tag{4}$$

where

$$w_x=\frac{1-j\sigma_x/(\varepsilon_0\omega)}{1-j\sigma_x^*/(\mu_0\omega)} \tag{5a}$$

$$w_y=\frac{1-j\sigma_y/(\varepsilon_0\omega)}{1-j\sigma_y^*/(\mu_0\omega)}. \tag{5b}$$

A general field component Ψ with magnitude Ψ_0 can be written as

$$\Psi=\Psi_0 e^{j\omega(t+x\sigma_x\sinh\varphi/(\varepsilon_0\omega c G)-y\cosh\varphi/(cG))} \times e^{(-\sinh\varphi(\omega/c))x}e^{-(\sigma_y\cosh\varphi/\varepsilon_0 cG)y}. \tag{6}$$

The second exponential term of (6) shows that the natural decay (i.e., the decay that is present even in the absence of σ_x and σ_y) in the positive x-direction is maintained. The first exponential term of (6) shows that an extra phase factor is induced by the conductivity σ_x. As for the (nonevanescent) y-direction, the usual damping due to σ_y is present in the third exponential term of (6). This proves that the PML-medium is not acting as an absorber for evanescent waves. The only damping that occurs is the inherent or natural damping that causes the evanescent behavior.

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III. Transmission Through a PML-PML Interface

Let's consider first the analytical case. It was proven by Berenger [1] that under certain conditions a perfect reflectionless transmission occurs at a PML-PML interface. Similar conclusions may be drawn for the evanescent case. At an interface normal to $x (x = 0)$, equalizing the exponents in (6) for the incident and the transmitted wave, gives

$$\left(1 - j\frac{\sigma_{y1}}{\varepsilon_0 \omega}\right)\frac{\cosh\varphi_1}{G_1} = \left(1 - j\frac{\sigma_{y2}}{\varepsilon_0 \omega}\right)\frac{\cosh\varphi_2}{G_2}. \quad (7)$$

Considering the continuity of the tangential components of both fields yields the reflection coefficient

$$r_p = \frac{E_{0r}}{E_{0i}} = \frac{G_1\cosh\varphi_2 - G_2\cosh\varphi_1}{G_1\cosh\varphi_2 + G_2\cosh\varphi_1}. \quad (8)$$

If the two media are matched, i.e., for $\frac{\sigma_i}{\varepsilon_0} = \frac{\sigma_i^*}{\mu_0}$ for $i = x1, y1, x2, y2$, and if $\sigma_{y1} = \sigma_{y2}$, then $G_1 = G_2 = 1$, and (7) reduces to $\varphi_1 = \varphi_2$, and $r_p = 0$. This conclusion is similar to what was obtained for propagating waves. However, since no additional damping occurs in the PML-medium, a small number of layers will generally not be sufficient to "absorb" the evanescent wave.

Moreover, extra numerical errors are induced by the actual implementation in a finite difference scheme. Without going into mathematical details, it is clear that the extra phase factor in the first exponential of (6) behaves as $\sinh\varphi = k_x/\omega$ and thus tends to infinity as ω tends to zero. Under these circumstances the numerical grid is not sampled enough to adequately represent the strong fluctuations of the field, which results in a large numerical error. The actual calculation of the reflection coefficient is slightly more complicated in the numerical case. We started by calculating the eigenmodes in both media that correspond to the plane wave solutions in the analytical case. We then used the continuity of the tangential electric field E_y at the interface and the finite difference equation for E_y itself as a set of boundary conditions (since the magnetic field is not available at the interface its continuity could not be demanded). This allowed us to calculate the exact numerical reflection produced by a layer of constant conductivity. The same technique was easily extended to other layer profiles, e.g., parabolic and geometrical [1], by using standard cascade theory.

IV. Numerical Examples

A two-dimensional parallel plate waveguide is terminated by a number of PML layers followed by an electric wall, and the waveguide is chosen sufficiently long to let the reflected field propagate freely toward its opposite end. A TM_{01}-mode is excited with a gaussian time-dependence just in front of the PML-medium. The reflection coefficient r_p is measured at the interface (this is important since r_p is very much dependent on location). This is done by performing a second experiment with no PML-medium present and with free propagation in both directions. This second experiment gives us the incident field E_{yi} at the interface, which is then subtracted from the total field E_{yt} obtained in the first experiment, in order to calculate the reflected field E_{yr}. After performing the Fourier

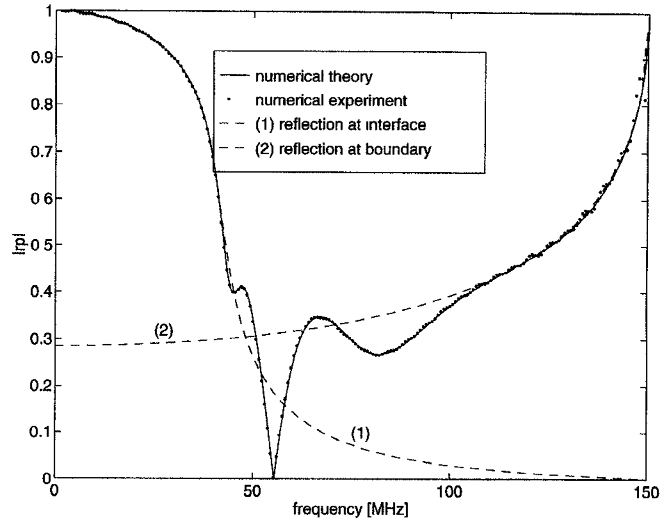

Fig. 1. Reflection coefficient below cutoff of a PML layer of constant conductivity.

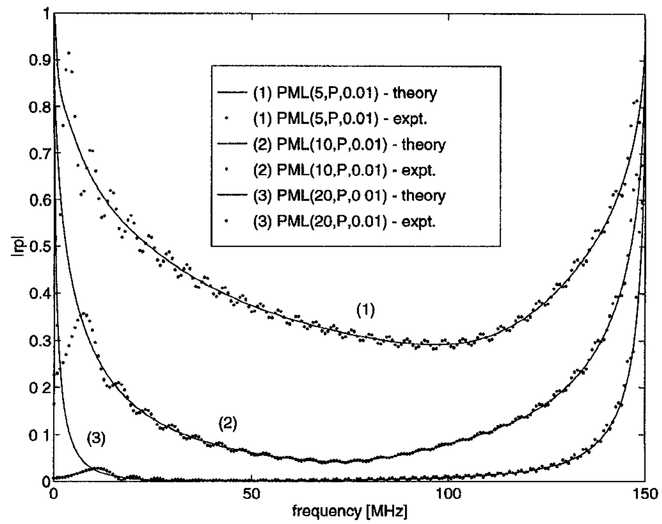

Fig. 2. Reflection coefficient below cutoff of PML layers with parabolic conductivity profile.

transform on the reflected field and the incident field, r_p is obtained as

$$r_p(\omega) = \frac{\tilde{E}_{yt}(\omega) - \tilde{E}_{yi}(\omega)}{\tilde{E}_{yi}(\omega)} \text{ with } \tilde{E}(\omega) = \text{FT}[E(t)]. \quad (9)$$

The width of the waveguide is 1 m with $\delta = \delta_x = \delta_y = 5$ cm and $\delta_t = 80$ ps. The gaussian pulse width is $T = 10\delta_t$, which is sufficient to encompass the frequency range from dc to the cut-off. Fig. 1 shows the amplitude of the reflection coefficient as a function of frequency for a layer of constant conductivity (PML(4,C,1) using the notation of [1]). Two curves are shown for comparison: 1) the reflection coefficient for the same PML-layer but with a free space termination at the right end instead of an electric wall in order to prevent the transmitted wave part from being reflected; and 2) the reflection coefficient for a free space layer of the same width as the PML-layer and terminated by an electric wall. The actual reflection coefficient is obviously a combination of both curves. In the low frequency region the numerical reflection is dominating and even tends to 1 in the static case. Near cut-off, however, the PML-medium is almost transparent and the reflection coefficient also goes to 1 as the decay length goes to infinity (at cut-off).

A second series of experiments was performed with the same waveguide for a more realistic parabolic conductivity profile. Fig. 2 shows the reflection coefficient for a PML-layer width of 5, 10, and 20 cells. The same conclusions may be drawn in this case.

V. Conclusion

Our analysis and numerical simulation show that the damping properties of the Berenger ABC do not extend to evanescent waves although the PML-medium theoretically still behaves as a reflectionless medium. In the low frequency range a considerable numerical error must be taken into account. This conclusion may be important to determine the width of the PML-layer in cases were the dominating wave part is evanescent, such as discontinuity problems in waveguides and certain scattering problems.

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